Developing Pedagogical Tools for Intervention: Approach, Methodology, and an Experimental Framework

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This paper reports on a project aimed at developing pedagogical tools for intervention in the number learning of low-attaining 3rd- and 4th-graders. Approaches to instructional design and intervention are described, and the use of the design research methodology is outlined. A major outcome of the project, an experimental framework for instruction, is described. The framework consists of five aspects: number words and numerals, structuring numbers to 20, conceptual place value, addition and subtraction to 100, and early multiplication and division. The descriptions of aspects include a discussion of low-attaining students' knowledge and difficulties, and details of instructional approaches developed in the project.

The Numeracy Intervention Research Project (NIRP) has the goal of developing pedagogical tools for intervention in the number learning of low-attaining third- and fourth-graders (8- to 10-year-olds). This paper reports on the NIRP by providing overviews of (a) the approach to intervention, (b) the use of a design research methodology, and (c) an experimental instructional framework consisting of five key aspects.

Approach to Intervention

A significant proportion of students have difficulties learning basic arithmetic (Louden et al., 2000). This limits their development of numeracy (McIntosh, Reys, & Reys, 1992; Yackel, 2001). Low-attainment is of particular concern in the context of the emphasis on numeracy, both nationally and internationally (e.g. *The national numeracy project: An HMI evaluation*, 1998; *Numeracy, a priority for all*, 2000). Furthermore, there are very few instructional programs to address numeracy difficulties and very few Australian schools systematically address this problem (Louden et al., 2000). Hence there are calls "to identify effective remedial approaches for the various identified weaknesses" (Bryant, Bryant, & Hammill, 2000, p. 174) and to develop approaches from the research-based reforms recommended for general mathematics education (Rivera, 1998). Researchers have developed programs of intervention in early number learning (Dowker, 2004; Gervasoni, 2005; Pearn & Hunting, 1995; Wright, Martland, Stafford, & Stanger, 2006; Young-Loveridge, 1991) focusing on topics such as counting and early addition and subtraction. The NIRP aims to extend this work with a focus on basic whole number arithmetic including reasoning with numbers in the hundreds and thousands, multidigit addition and subtraction, and early multiplication and division.

Organising by key aspects. Recent intervention programs have described early number knowledge in terms of components (Dowker, 2004) and domains (Clarke, McDonough, & Sullivan, 2002). These descriptions highlight the idiosyncratic nature of students' number knowledge (Gervasoni, 2005) and learning paths (Denvir & Brown, 1986). In this paper we use a framework approach (Wright et al., 2006) to set out five key aspects of number learning that we regard as important for intervention with 3rd- and 4th-graders. Our approach involves developing instructional activities relevant to each of these key aspects. In this approach, constructing a framework of key aspects is important in developing a domain-specific theory for intervention instruction. Further, this framework can be applied to all students and can inform classroom instruction.

Instructional design. Progressive mathematisation refers to the development from informal, context-bound thinking to more formal thinking (Beishuizen & Anghileri, 1998; Gravemeijer, 1997; Treffers, 1991). As in the emergent modelling heuristic (Gravemeijer, Cobb, Bowers, & Whitenack, 2000), instructional design involves anticipating a potential learning trajectory, and devising an instructional sequence of tasks which foster students' progressive mathematisation along the trajectory, through levels of thinking from informal to formal. Particular settings, such as manipulative equipment or notation systems, can have an important role in an instructional sequence. A setting can be established as a context for students' initial context-dependent thinking, and then become a *model for* more independent numerical reasoning, thus mediating the crucial development from concrete toward more abstract thinking (Gravemeijer, 1997). An instructional sequence consists of *instructional procedures*, each of which serves to incrementally distance the student from the materials, advance the complexity of the task, and potentially raise the sophistication of the student's thinking. Detailed assessment of the student's knowledge informs the teacher's selection of instructional procedures. Instruction focuses on engaging the student in independent, sustained thinking, and observational assessment enables tuning instruction to the cutting edge of the student's knowledge (Wright et al., 2006).

Approach to number instruction. Our approach to instruction emphasises flexible, efficient computation, and strong numerical reasoning (Beishuizen & Anghileri, 1998; Heirdsfield, 2001; Yackel, 2001). Mental computation, in particular, is foundational for efficient computation, numerical reasoning, and number sense (McIntosh et al., 1992; Treffers, 1991). Learning builds from students' own informal mental strategies (Beishuizen & Anghileri, 1998; Gravemeijer, 1997). However, students need to develop flexible, efficient, mathematically sophisticated strategies. Low-attaining students often use inefficient count-by-ones strategies, and error-prone rote procedures, and depend on supporting materials or fingers (Gray, Pitta, & Tall, 2000; Wright, 2001). Hence, intervention instruction needs to develop students' number knowledge to support non-count-by-ones strategies, and to move students to independence from materials.

Methodology

The NIRP adopted a methodology based on design research (Cobb, 2003; Gravemeijer, 1994), consisting of three one-year design cycles. The NIRP aimed to develop pedagogical tools for intervention, consisting of a framework, assessment tasks, and instructional sequences. Each design cycle consisted of (a) initial development of the pedagogical tools, (b) use of the tools in an intervention program with teachers and students, (c) analysis of the learning and teaching in the program, and (d) refinement of the tools based on the analysis. Within each cycle, analysis and development were on-going, in meetings of the researchers and project teachers, in analysis of assessments, and in teachers' daily lesson planning. The analysis of the learning and teaching in the intervention program is informed by a teaching experiment methodology (Steffe & Thompson, 2000). Interview assessments and instructional sessions were videotaped, providing an extensive empirical base for analysis. The approach to the development of intervention programs described in this paper is an appropriate response to Ginsburg's (1998) call for teaching experiments focusing on students with learning difficulties.

The Study

The intervention program for each year involved eight or nine teachers, each from a different school, across the state of Victoria. In each school, 12 students were identified as low-attaining in arithmetic, based on screening tests administered to all third- and fourth-graders. In each school (a) in term 2, these 12 students were assessed in individual interviews; (b) in term 3, eight of the low-attaining students participated in intervention teaching cycles; and (c) in term 4, the 12 students were again assessed in individual interviews. The teaching cycles involved teaching sessions of 30 minutes duration, for four days per week, for 10 weeks. Two students were taught as singletons and six as trios, and all of the instructional sessions with singletons were videotaped. Across the three years of the project, in each of 25 schools, the project teacher assessed 300 low-attaining students, each on two occasions, taught 50 students individually and 150 students in trios.

Development of the Instructional Framework

Through the cycles of design research, the framework of key aspects of knowledge developed from four considerations. Firstly, areas of significance were identified in our analysis of low-attaining students' number knowledge and difficulties, areas that seem to be characteristic of what successful students can do and what low-attaining students cannot do. Secondly, these areas were clarified in making a coherent framework for teachers to use for analysing assessments and profiling students' learning needs. Thirdly, the key aspects became further defined as the key instructional sequences and their associated settings emerged. Fourthly, the key aspects were refined in articulating a coherent framework for instruction. The framework is experimental in the design research sense – it is intended to be further trialled, analysed, and developed.

The resulting instructional framework consists of the following five aspects: (A) Number Words and Numerals; (B) Structuring Numbers 1 to 20; (C) Conceptual Place Value; (D) Addition and Subtraction 1 to 100; and (E) Early Multiplication and Division.

Experimental Instructional Framework

For each of aspects A to D, we describe (a) the significance of the aspect; (b) low-attaining students' knowledge and difficulties; and (c) instructional sequences. Due to space limitations, aspect E is not described in this paper.

Aspect A: Number Words and Numerals

Low-attaining students' knowledge and difficulties. Early number curricula focus on number word sequences (NWS) to 20, and to 100, and learning to read and write 2-digit numerals. Students' early difficulties are well-documented (e.g., Fuson, Richards, & Briars, 1982). Classroom instruction on NWSs and numerals tends to decrease as students progress through school. However, low-attaining third- and fourth-graders have significant difficulties with these areas (Hewitt & Brown, 1998). Errors with NWSs in the range 1 to 100 include: (a) “52, 51, 40, 49, 48…” and (b) “52, 51, 49, 48…”. Students are aware of the chains of number words in each decade – 41 to 49 and 51 to 59, and link these chains incorrectly when going backwards (Skwarchuk & Anglin, 2002). Errors with NWSs in the range 100 to 1000 occur at decade and hundred numbers, for example: (a) “108, 109, 200, 201, 202…”; (b) “198, 199, 1000, 1001…”; and (c) “202, 201, 199, 198…” (Ellemor-Collins & Wright, in press). When students respond correctly on these tasks, in many cases they lack certitude. Knowledge of sequences of tens off the decade is an important part of knowledge of base-ten structures (Ellemor-Collins & Wright, in press), and is a prerequisite for mental jump strategies (Fuson et al., 1997; Menne, 2001). Some low-attainers cannot skip count by tens off the decade. Given the task “Count by tens from 24”, responses included: (a) “24, 25, 20”; (b) “24, 30, 34, 40”; (c) “24…34…44” with each ten counted by ones subvocally; and (d) “I can’t do that”. As well, there is a range of significant errors with sequences of tens beyond 100 (Ellemor-Collins & Wright, in press). Some 3rd- and 4th-graders make errors with 3-digit and 4-digit numerals involving zeros (Hewitt & Brown, 1998): 306 is identified as “360”; 6032 is identified as “6 hundred and 32”, or "60 thousand and 32”; and 1005 is written “10 005”.

Instruction in number words and numerals. Facility with number word sequences and numerals is important, and requires explicit attention for low-attainers (Menne, 2001; Wright et al., 2006). This aspect includes identifying and writing numerals to 1000 and beyond. Instruction focuses on reciting and reasoning with number word sequences and numeral sequences, without structured settings such as number lines or base-ten materials. Students develop knowledge of the auditory and visual patterns, somewhat separate from numerical reasoning about quantity and position (Hewitt & Brown, 1998; Skwarchuk & Anglin, 2002). We have found that explicit instruction focusing on bridging 10s, 100s and 1000s, forwards and backwards, is productive. Saying sequences by tens and hundreds, on and off the decade, supports development of place value knowledge. Saying sequences by 2s, 3s, and 5s, on and off the multiple, supports development of multiplicative knowledge. Students can and should learn number word sequences and numerals in a number range well in advance of learning to add and subtract in that range because familiarity with a range of numbers establishes a basis for meaningful arithmetic (Wigley, 1997).

Exemplar instructional sequence: the numeral track. The numeral track is an instructional device consisting of a sequence of ten numerals, each of which is adjacent to a lid which can be used to conceal the numeral (Wright et al., 2006). In the instructional sequence, first the lids are opened sequentially, and the student names each numeral in turn, after seeing the numeral. Second, when the sequence is familiar, the student’s task is to name each numeral in turn, before seeing the numeral. In this case, the opening lids enable self-verification. Third, the number sequence can be worked backwards. Finally, more advanced tasks can be used. For example, one lid is opened and the teacher points to other lids for the student to name: the number before, the number two after, and so on. In this setting, learning about NWSs supports and is supported by learning about sequences of numerals. The teacher selects the sequence: bridging 110, a tens sequence off the decade, a 2s sequence, and so on. The teacher can observe a student’s specific difficulty, and finely adjust the instructional tasks. The lids allow incremental distancing from the material and internalisation of the sequence.

Aspect B: Structuring Numbers 1 to 20

Facile calculation in the range 1 to 20. Learning arithmetic begins with learning to add and subtract in the range 1 to 20. Students' initial strategies involve counting-by-ones (Fuson, 1988; Steffe & Cobb, 1988) and developing this facility is an important aspect of early number learning. Students can then develop strategies more sophisticated than counting by ones, such as adding through ten (eg., 6 + 8 = 8 + 2 + 4), using fives (6 + 7 = 5 + 5 + 1 + 2), and near-doubles (eg., 6 + 7 = 6 + 6 + 1). Developing these strategies builds on knowledge of combining and partitioning numbers (Bobis, 1996; Gravemeijer et al., 2000; Treffers, 1991). Efficient calculation also involves knowledge of additive number relations, such as commutativity (8 + 9 = 9 + 8), and inversion (15 + 2 = 17 implies that 17 – 15 = 2). The development of efficient, non-count-by-ones calculation in the range 1 to 20 is important. Counting-by-ones can be slow, and error-prone. Further, facile calculation promotes number sense and numerical reasoning (Treffers, 1991), and develops a part-whole conception of numbers (Steffe & Cobb, 1988), providing a basis for further learning such as the construction of units of 10 and multiplicative units (Cobb & Wheatley, 1988).

Low-attaining students' knowledge and difficulties. Low-attaining 3rd- and 4th-graders typically will solve addition and subtraction tasks in the range 1 to 20 by counting on and counting back (Gray et al., 2000; Wright, 2001). As well, they will not necessarily use the more efficient counting strategy, solving 17 – 15 for example, by making 15 counts back from 17 and keeping track on their fingers. They do not seem to partition numbers spontaneously when attempting to add or subtract. These students typically have difficulty with tasks such as stating two numbers that add up to 19. They might know all or most doubles in the range 1 to 20, but will not use a double to work out a near-double addition (6 + 7). As well, they might solve without counting, addition tasks with 10 as the first addend (10 + 5) but will not apply the ten structure of teen numbers (14 is 10 + 4) to solve addition (14 + 4) or subtraction (15 – 4), and will not use adding through 10 to solve tasks such as 9 + 5. This preference for counting-by-ones has been explained as a preference to think procedurally (Gray et al., 2000).

Instruction in structuring numbers 1 to 20. The arithmetic rack (Treffers, 1991) is an important instructional device, enabling flexible patterning of the numbers 1 to 20 in terms of doubles, five, and ten. Instruction proceeds in three phases: (a) making and reading numbers; (b) addition involving two numbers; and (c) subtraction involving two numbers (Wright et al., 2006). In each phase, the teacher advances the complexity, from tasks with smaller numbers and more familiar structures, to tasks with larger numbers and less familiar structures. In each phase, the teacher also uses screening and flashing to progressively distance the student from the setting. The student is actively reasoning, in the context of the structured patterns. The intention is that activity with the rack is increasingly internalised and the student shifts from reasoning with numbers as referents-to-the-beads, to numbers as independent entities (Gravemeijer et al., 2000). Instruction with the arithmetic rack can overcome low-attainers' reticence to relinquish counting-by-one strategies.

Aspect C: Conceptual Place Value

Multidigit knowledge. Research evidence supports building multidigit arithmetic on students' informal understandings of number, and emphasizing mental strategies with 2-digit numbers (Beishuizen & Anghileri, 1998; Fuson et al., 1997; Yackel, 2001). Efficient mental strategies require sound knowledge of structures in multidigit numbers such as: (a) additive place value (25 is 20 and 5); (b) jumping by ten, on and off the decade (40 + 20 = 60, 48 + 20 = 68); (c) jumping within and across decades (68 + 5 = 68 + 2 + 3 = 73); and (d) locating neighbouring decuples (linking 48 + 25 to 50 + 25) (Ellemor-Collins & Wright, in press; Heirdsfield, 2001; Menne, 2001; Yackel, 2001). These structures are based on the decade patterns and units of ten. Other important structures include doubles and halves: double 25 is 50, double 50 is 100. Together these structures form a rich network of number relations, the basis of flexible and efficient computation (Foxman & Beishuizen, 2002; Heirdsfield, 2001; Threlfall, 2002). Instruction on these multidigit structures can be distinguished from formal place value instruction. Thompson and Bramald (2002), for example, observe that students' intuitive strategies depend on quantity value, the informal additive aspect of place value, which they distinguish from column value, the formal written aspect of place value. Place value tasks involving manipulation of numerals and knowing column value are problematic for many students, especially low-attainers (Beishuizen & Anghileri, 1998; Thompson & Bramald, 2002). Younger students reason about numbers first in terms of verbal sequences and quantities, rather than written symbols, so addition by formal manipulations of symbols is not necessarily meaningful for these students (Cobb & Wheatley, 1988; Treffers, 1991). For example, a student might understand the result of jumping by ten, but not of adding one in the tens column. Where regular place value instruction is intended to support the development of standard, written algorithms, we propose conceptual place value as an approach to support the development of students' intuitive arithmetical strategies.

Low-attaining students' knowledge and difficulties. Low-attaining third- and fourth-graders typically will not increment or decrement by ten off the decade when solving 2-digit addition and subtraction tasks. In a task presenting, with base-ten materials, 48 + 2 tens and 5 ones, some low-attainers find the total by counting by ones from 48. Other students will attempt to use a split strategy (40 + 20 and 8 + 5) to solve these tasks but will have difficulty recombining tens and ones (Cobb & Wheatley, 1988). These students either lack place value knowledge or are unable to use place value knowledge in dynamic situations, that is, situations that involve increasing or decreasing numbers by ones, tens or hundreds. We regard these difficulties as symptomatic of a lack of important knowledge about multidigit numbers (Ellemor-Collins & Wright, in press).

Instruction in conceptual place value. Conceptual place value encompasses instructional sequences that develop knowledge of the structure of multidigit numbers, as a foundation for mental computation. The main instructional sequence involves flexibly incrementing and decrementing by ones and tens, and later hundreds and thousands, in the context of base-ten materials. Two important settings are: (a) bundling sticks and (b) dots on laminated card organised into ten strips and hundred squares. These seem to be more authentic and hence more useful than MAB blocks. Instructional tasks include firstly, building 2-digit numbers, and then incrementing and decrementing by one ten, two tens, one ten and two ones, and so on. The teacher incrementally distances the student from the setting. Initially, the material is visible. The student answers, and then might reorganise the tens and ones to verify their answer. As the instructional sequence develops, the material is screened and the screens are removed to enable verification. This instruction elicits reasoning about quantities in the range 1 to 100, thus providing a basis for 2-digit addition and subtraction using jump strategies (aspect D). As well, this instruction is extended to flexibly incrementing and decrementing 3- and 4-digit numbers, by ones, tens and hundreds. In this way, students' first learning of place value is strongly verbal and occurs in an additive sense. We have also found that Arrow Cards (Hewitt & Brown, 1998; Wigley, 1997) can be very useful in further supporting this learning.

Aspect D: Addition and Subtraction to 100

Flexible, efficient multidigit computation. Developing facile mental strategies for addition and subtraction involving two 2-digit numbers is a critically important goal of arithmetic learning in the first three or four years of school. This lays a strong foundation for all further learning of arithmetic, including multiplication and division, fractions and decimals, and so on. As well, strong mental strategies will support learning of the standard written algorithms and efficient use of calculators in mathematical problem solving (Beishuizen & Anghileri, 1998). Two main categories of efficient strategies are jump strategies and split strategies. Variations and alternatives abound. (Foxman & Beishuizen, 2002; Fuson et al., 1997; Klein, Beishuizen, & Treffers, 1998; Thompson & Bramald, 2002). All these strategies involve jumping in tens and jumping through ten. Jumping through ten can be used for example, to solve 68 + 7 as 68 + 2 + 5, and more generally involves adding and subtracting to and from a decuple (60 + 8, 47 + x = 50, 74 – x = 70, 60 – 4).

Low-attaining students' knowledge and difficulties. As with tasks involving base-ten materials described in aspect C above, some low-attaining third- and fourth-graders seem to interpret written tasks such as 38 + 24 and 63 – 24 as an instruction to make 24 counts forwards or backwards respectively (Wright, 2001). Also, low-attainers frequently try to use a split strategy for written tasks, but have difficulty recombining tens and ones (Beishuizen, Van Putten, & Van Mulken, 1997; Fuson et al., 1997). As well, when solving a task such as 46 + 53, by adding 40 and 50 and 6 and 3 (split strategy), they will typically count-on to work out each of the two sums (40 + 50 and 6 + 3). These students do not know about jumping in tens and jumping through ten to add or subtract in the range 1 to 100 (Menne, 2001). Research suggests that most successful students use jump strategies, whereas most low-attainers use split strategies; further, low-attainers who do use jump have more success and flexibility than those who use split (Beishuizen et al., 1997; Foxman & Beishuizen, 2002; Klein et al., 1998). As well, students who have been taught place value in the traditional way, are likely to have a preference for split strategies.

Instruction in addition and subtraction to 100. Incrementing and decrementing by ten is one important prerequisite for learning to use jump strategies in the range 1 to 100. A second is having facile strategies for addition and subtraction in the range 1 to 20 (Menne, 2001). Our experience is that low-attainers who are facile in the range 1 to 20 require explicit instruction in applying this knowledge when adding and subtracting two 2-digit numbers. For this instruction we have found it useful to use ten frame cards in two forms – a ten frame card for each of the numbers 1 to 9, and full ten frame cards for the decuples. In this setting, 38 can be shown using 3 ten-cards and one eight-card. The ten frame cards used in this way, can supports students' reasoning about adding and subtracting to and from a decuple. This approach can be extended firstly to addition and subtraction involving a 1-digit and a 2-digit number (64 + 3, 78 + 6, 47 – 4, 82 – 7) and finally to addition and subtraction involving two 2-digit numbers. We use a notation system in conjunction with mental strategies. The notation is used to record the mental strategy rather than providing a means of solving the task. Notation supports reflection and communication, and is important for increasing robustness, curtailment and flexibility (Gravemeijer et al., 2000; Klein et al., 1998). We have found four notation systems useful. The empty number line notation (Klein et al., 1998) is used for jump and related strategies. Also used for jump strategies is the simple arrow notation (48 + 25, 48→50, 50→70, 70→73). The so-called drop-down notation is used for split strategies and notation involving a progression of number sentences (arithmetical equations) can be used for either jump or split strategies.

Conclusion

An important intention of the framework is to bring together aspects of number variously identified as areas where low-attaining students do not progress. A second important intention is to bring together powerful instructional sequences specific to each of those aspects. The consistent approach to instructional design in terms of progressive mathematisation promotes coherence across the framework. Further, by and large, instruction in the aspects proceeds concurrently, and the teacher makes connections between the aspects (Treffers, 1991). The goal, overall, is the coherent development of students' facility with whole number arithmetic.

The experimental framework initiates further lines of inquiry at four levels: (a) analyse further, low-attaining students' learning within each aspect; (b) refine the instructional sequences and their connections; (c) assess students' and teachers' responses to intervention programs based on the framework; (d) Clarify the design research approach to developing pedagogical materials for intervention.

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